

FRACTAL RANDOM SERIES GENERATED BY POISSON-VORONOI TESSELLATIONS

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ABSTRACT. In this paper, we construct a new family of random series defined on \mathbb{R}^D , indexed by one scaling parameter and two Hurst-like exponents. The model is close to Takagi-Knopp functions, save for the fact that the underlying partitions of \mathbb{R}^D are not the usual dyadic meshes but random Voronoi tessellations generated by Poisson point processes. This approach leads us to a continuous function whose random graph is shown to be fractal with explicit and equal box and Hausdorff dimensions. The proof of this main result is based on several new distributional properties of the Poisson-Voronoi tessellation on the one hand, an estimate of the oscillations of the function coupled with an application of a Frostman-type lemma on the other hand. Finally, we introduce two related models and provide in particular a box-dimension calculation for a derived deterministic Takagi-Knopp series with hexagonal bases.

INTRODUCTION

The original Weierstrass series (see [38]) is one of the most well-known examples of continuous but nowhere differentiable functions. Among the more general family of Weierstrass-type functions, the Takagi-Knopp series can be defined in one dimension as

$$\forall x \in \mathbb{R}, \quad K_H(x) = \sum_{n=0}^{\infty} 2^{-nH} \Delta(2^n x) \quad (1)$$

where $\Delta(x) = \text{dist}(x, \mathbb{Z})$ is the sawtooth -or pyramidal- function and $H \in (0, 1]$ is called the Hurst parameter of the signal. Introduced at the early beginning of the 20th century (see [34, 21]), they have been extensively studied since then (see the two recent surveys [2, 23]).

The construction of K_H is only based on two ingredients: a sequence of partitions of \mathbb{R} (the dyadic meshes) associated with a decreasing sequence of amplitudes for the consecutive layers of pyramids. Therefore, we can easily extend definition (1) to dimension $D \geq 2$ using the D -dimensional dyadic meshes.

In order to provide realistic models, it is needed to randomize such deterministic functions. Two common ways to do it are the following: either the pyramids are translated at each step by a random vector (see e.g. [36, 16, 11]), or the height of each pyramid is randomly chosen (see in particular [15] for the famous construction of the Brownian bridge). We obtain suitable models for highly irregular signals such as rough surfaces (see [15, 27, 12] and [32] Chapter 6).

In many cases, the graph of such functions is a fractal set. Therefore, their fractal dimensions constitute a crucial information for estimating the roughness of the data. The two most

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common fractal dimensions are the box-dimension and the Hausdorff dimension. The former is in general easier to get whereas the latter is known only in very special cases (see e.g. [26, 24, 19, 9, 4]).

In this paper, a new family of Takagi-Knopp type series is introduced. Contrary to the previous randomization procedures, our key-idea is to substitute a sequence of random partitions of \mathbb{R}^D for the dyadic meshes. An alternative idea would have been to keep the cubes and choose independently and uniformly in each cube each center of a pyramid. Notably because the mesh has only D directions, it would be very tricky to calculate a Hausdorff dimension. One advantage for applicational purposes may be to get rid of the rigid structure induced by the cubes and to provide more flexibility with the irregular pattern. A classical model of random partition is the Poisson-Voronoi tessellation.

For a locally finite set of points called nuclei, we construct the associated Voronoi partition of \mathbb{R}^D by associating to each nucleus c its cell \mathcal{C}_c , i.e. the set of points which are closer to c than to any other nucleus. When the set of nuclei is a homogeneous Poisson point process, we speak of a Poisson-Voronoi tessellation [30, 28, 7]. In particular, it is invariant under any transformation of \mathbb{R}^D which preserves the Lebesgue measure and its cells are almost surely convex polytopes. Classical results include limit theorems [1], distributional [5, 6] and asymptotic results [18, 8] for the so-called typical Poisson-Voronoi cell. The model is commonly used in various domains such as molecular biology [31], thermal conductivity [22] or telecommunications (see e.g. [37] and Volume 1, Chapter 5 from [3]). The only parameter needed to describe the tessellation is the intensity $\lambda > 0$, i.e. the mean number of nuclei or cells per unit volume. In particular, the mean area of a typical cell from the tessellation is λ^{-1} . Multiplied by the scaling factor $\lambda^{\frac{1}{D}}$, the Poisson-Voronoi tessellation of intensity λ is equal in distribution to the Poisson-Voronoi tessellation of intensity one. This scaling invariance is a crucial property that will be widely used in the sequel.

Let $\lambda > 1$ and $\alpha, \beta > 0$. The parameter λ is roughly speaking a scaling factor and α, β are Hurst-like exponents. For every $n \geq 0$, we denote by \mathcal{X}_n an homogeneous Poisson point process of intensity $\lambda^{n\beta}$ in \mathbb{R}^D and by $\mathcal{T}_n = \{\mathcal{C}_c : c \in \mathcal{X}_n\}$ the set of cells of the underlying Poisson-Voronoi tessellation. We recall that $\lambda^{\frac{n\beta}{D}} \mathcal{T}_n \stackrel{\text{def}}{=} \{\lambda^{\frac{n\beta}{D}} \mathcal{C}_c : c \in \mathcal{X}_n\}$ is distributed as \mathcal{T}_0 and $\lambda^{\frac{\beta}{D}} \mathcal{T}_n \stackrel{\text{law}}{=} \mathcal{T}_{n-1}$ thanks to the scaling invariance.

Let $\Delta_n : \mathbb{R}^D \rightarrow [0, 1]$ be the random pyramidal function satisfying $\Delta_n = 0$ on $\bigcup_{c \in \mathcal{X}_n} \partial \mathcal{C}_c$ and $\Delta_n = 1$ on \mathcal{X}_n (see Figure 1).

In particular we can then define a continuous function by

$$\forall x \in \mathbb{R}^D, \quad F_{\lambda, \alpha, \beta}(x) = \sum_{n=0}^{\infty} \lambda^{-\frac{n\alpha}{D}} \Delta_n(x). \quad (2)$$

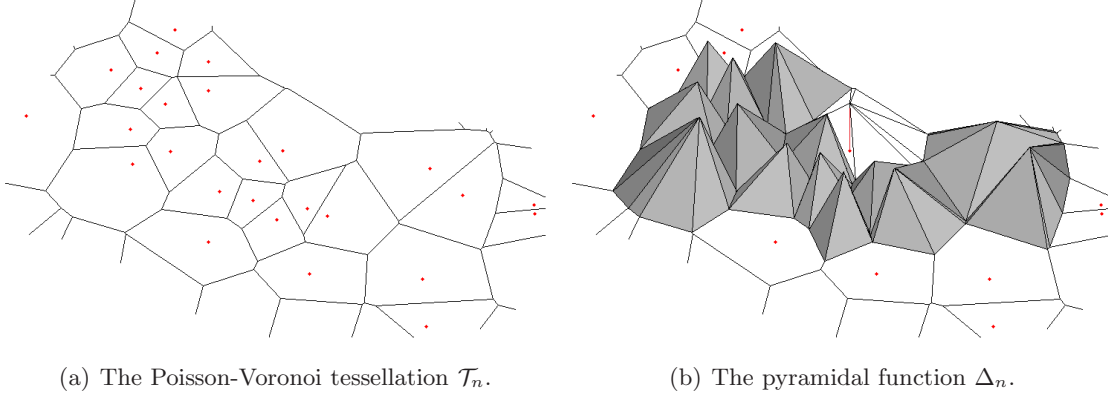
Let us denote by $\dim_B(K)$ and $\dim_H(K)$ the (upper) box-dimension and the Hausdorff dimension of a non-empty compact set K (see e.g. [14] for precise definitions). We are mainly interested in the exact values of these dimensions. Our result is the following:

Theorem 1. *Let $\lambda > 1$ and $0 < \alpha \leq \beta \leq 1$. Then $F_{\lambda, \alpha, \beta}$ is a continuous function whose random graph*

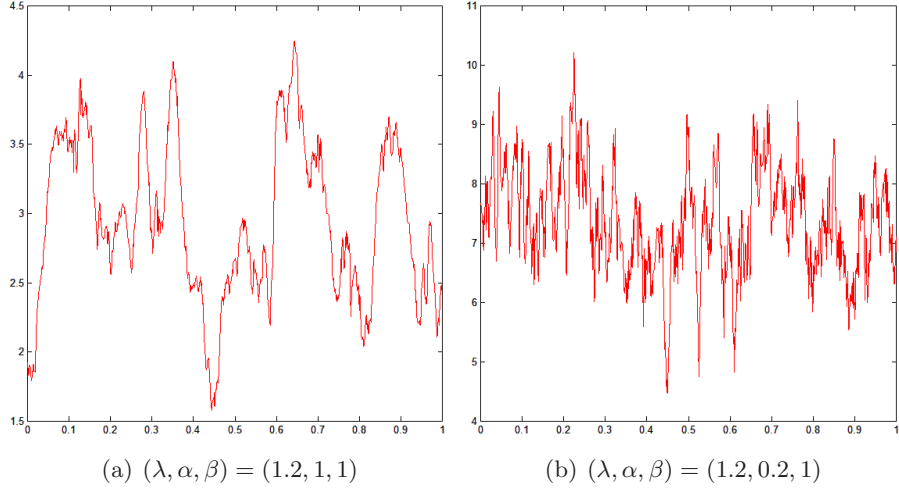
$$\Gamma_{\lambda, \alpha, \beta} = \{(x, F_{\lambda, \alpha, \beta}(x)) : x \in [0, 1]^D\} \subset \mathbb{R}^D \times \mathbb{R}$$

is a fractal set satisfying almost surely

$$\dim_B(\Gamma_{\lambda, \alpha, \beta}) = \dim_H(\Gamma_{\lambda, \alpha, \beta}) = D + 1 - \frac{\alpha}{\beta}. \quad (3)$$

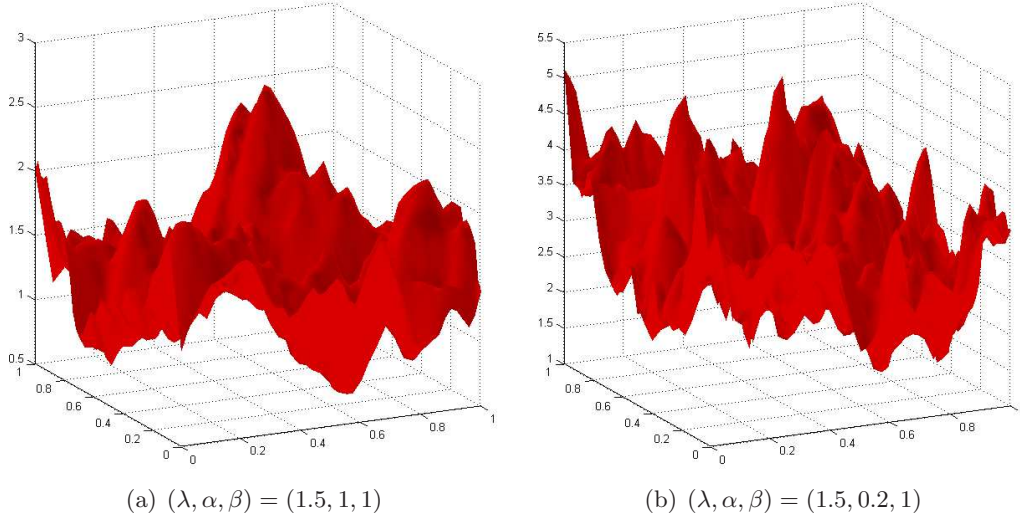

 FIGURE 1. Construction of the elementary piecewise linear function Δ_n .

Equalities (3) imply that the smaller $\frac{\alpha}{\beta}$ is, the more irregular $F_{\lambda,\alpha,\beta}$ and $\Gamma_{\lambda,\alpha,\beta}$ are (see Figure 2 and Figure 3). The result of Theorem 1 naturally holds when $[0, 1]^D$ is replaced with any cube of \mathbb{R}^D .


 FIGURE 2. Graph of the random function $F_{\lambda,\alpha,\beta}$ when $D = 1$.

The paper is organized as follows. In the first section we state some preliminary results related to the geometry of the Poisson-Voronoi tessellations. We introduce in particular the oscillation sets $\mathcal{O}_{n,N}$ (see (6)) that are used to derive explicit distributional properties on the increments of Δ_n and precise estimates on the increments of $F_{\lambda,\alpha,\beta}$. Section 2 is then devoted to the proof of Theorem 1, i.e. an upper bound for $\dim_B(\Gamma_{\lambda,\alpha,\beta})$ comes from the estimation of the oscillations of $F_{\lambda,\alpha,\beta}$ whereas a lower bound for $\dim_H(\Gamma_{\lambda,\alpha,\beta})$ is obtained via a Frostman-type lemma. Finally, we introduce and study in the last section two related models: a deterministic series based on an hexagonal mesh and a random series based on a perturbation of the dyadic mesh.

In the sequel we will drop the indices λ , α and β so that $F = F_{\lambda,\alpha,\beta}$ and $\Gamma = \Gamma_{\lambda,\alpha,\beta}$.

FIGURE 3. Graph of the random function $F_{\lambda, \alpha, \beta}$ when $D = 2$.

1. PRELIMINARY RESULTS

1.1. Notations.

We consider the metric space \mathbb{R}^D , $D \geq 1$, endowed with the Euclidean norm $\|\cdot\|$. The closed ball with center $x \in \mathbb{R}^D$ and radius $r > 0$ is denoted by $B_r(x)$. We write $\text{Vol}(A)$ for the Lebesgue measure of a Borel set $A \subset \mathbb{R}^D$. In particular $\kappa_D = \text{Vol}(B_1(0))$. The unit sphere of \mathbb{R}^D is denoted by \mathbb{S}^{D-1} and σ_{D-1} will be the unnormalized area measure of \mathbb{S}^{D-1} . The area surface of \mathbb{S}^{D-1} is then $\omega_{D-1} = \sigma_{D-1}(\mathbb{S}^{D-1})$. Finally, for all $s \geq 0$, the s -dimensional Hausdorff measure is \mathcal{H}^s .

For all $x, y \in [0, 1]^D$ and all $n \geq 0$ let

$$Z_n(x, y) = \lambda^{-\frac{n\alpha}{D}} (\Delta_n(x) - \Delta_n(y)) \quad (4)$$

so that $F(x) - F(y) = \sum_{n=0}^{\infty} Z_n(x, y)$, and

$$S_n(x, y) = \sum_{\substack{m=0 \\ m \neq n}}^{\infty} Z_m(x, y) \quad (5)$$

so that $F(x) - F(y) = Z_n(x, y) + S_n(x, y)$. Notice that $Z_n(x, y)$ and $Z_m(x, y)$ are two independent random variables for $m \neq n$. In particular $Z_n(x, y)$ and $S_n(x, y)$ are independent.

Finally, we fix $H > \beta$. For all $n \geq 0$, we set $\tau_n = \lambda^{-\frac{nH}{D}}$.

1.2. Random oscillations sets.

Remember that the function Δ_n is locally linear. Any maximal set on which Δ_n is linear is the convex hull $\text{Conv}(\{c\} \cup f)$ of the union of a nucleus c from \mathcal{X}_n and a hyperface f of the cell associated with c . The set \mathcal{S}_n of such simplices tessellates \mathbb{R}^D . For all $n, N \geq 0$ we define the random sets

$$\mathcal{O}_{n,N} = \{x \in [0, 1]^D : \text{all points of } B_{\tau_N}(x) \text{ are on the same simplex of } \mathcal{S}_n \text{ as } x\} \quad (6)$$

and

$$W_N = \bigcap_{n=N}^{\infty} \mathcal{O}_{n,n}. \quad (7)$$

The first result states that these sets are not too ‘small’.

Proposition 1.1.

(i) *There exists a constant $C > 0$ such that, for all $x \in [0, 1]^D$ and all $n, N \geq 0$,*

$$\mathbb{P}(x \notin \mathcal{O}_{n,N}) \leq C \lambda^{\frac{n\beta - NH}{D}}.$$

(ii) *We have $\lim_{N \rightarrow \infty} \mathbb{P}(\text{Vol}(W_N) > 0) = 1$.*

Proof.

(i) By invariance by translation of \mathcal{X}_n and \mathcal{T}_n , we notice that for every $x \in \mathbb{R}^D$,

$$\mathbb{P}(0 \notin \mathcal{O}_{n,N}) = \mathbb{P}(x \notin \mathcal{O}_{n,N}). \quad (8)$$

Let Sk_n be the skeleton of the simplex tessellation \mathcal{S}_n , i.e. the union of the boundaries of all simplices (see the grey region on Figure 4).

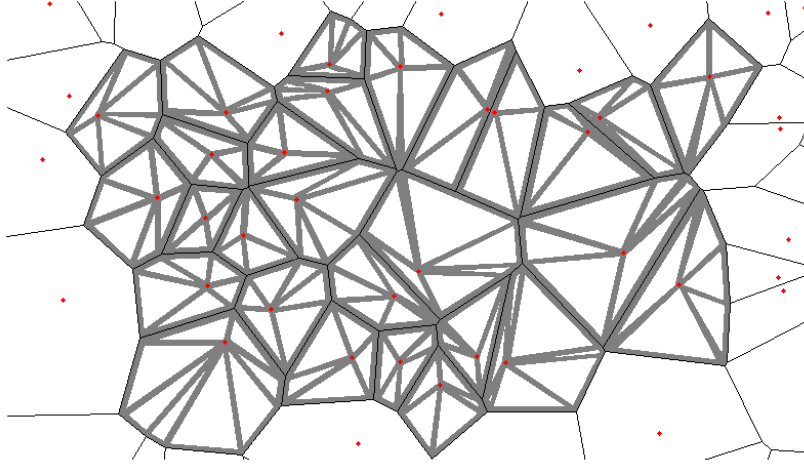


FIGURE 4. The skeleton of the complete tessellation (in grey).

In particular, we have the equivalence

$$x \notin \mathcal{O}_{n,N} \iff x \in \text{Sk}_n + B_{\tau_N}(0). \quad (9)$$

Let U be a uniform point in $[0, 1]^D$, independent of the tessellation \mathcal{T}_n , and $\mathbb{P}_{\mathcal{X}_n}$ be the distribution of the Poisson point process \mathcal{X}_n . Using (8) and Fubini’s theorem, we get

$$\mathbb{P}(U \notin \mathcal{O}_{n,N}) = \int_{[0,1]^D} \mathbb{P}_{\mathcal{X}_n}(x \notin \mathcal{O}_{n,N}) dx = \mathbb{P}(0 \notin \mathcal{O}_{n,N}).$$

Moreover, the equivalence (9) implies that

$$\mathbb{P}(0 \notin \mathcal{O}_{n,N}) = \mathbb{E}_{\mathcal{X}_n} \left(\int_{[0,1]^D} \mathbb{I}_{\text{Sk}_n + B_{\tau_N}(0)}(x) dx \right) = \mathbb{E}_{\mathcal{X}_n} (\text{Vol}((\text{Sk}_n + B_{\tau_N}(0)) \cap [0, 1]^D)). \quad (10)$$

It remains to calculate the Lebesgue measure of the set $(\text{Sk}_n + B_{\tau_N}(0)) \cap [0, 1]^D$. Denoting by \mathcal{F}_n the set of hyperfaces of the simplex tessellation \mathcal{S}_n which intersect $[0, 1]^D$ we have

$$\begin{aligned} \text{Vol}((\text{Sk}_n + B_{\tau_N}(0)) \cap [0, 1]^D) &\leq \sum_{f \in \mathcal{F}_n} \text{Vol}((f + B_{\tau_N}(0)) \cap [0, 1]^D) + \text{Vol}(\partial([0, 1]^D) + B_{\tau_N}(0)) \\ &\leq 2\lambda^{-\frac{NH}{D}} \left(2D + \sum_{f \in \mathcal{F}_n} \mathcal{H}^{D-1}(f \cap [0, 1]^D) \right). \end{aligned}$$

Consequently, we have

$$\mathbb{E}_{\mathcal{X}_n} (\text{Vol}((\text{Sk}_n + B_{\tau_N}(0)) \cap [0, 1]^D)) \leq 2\lambda^{-\frac{NH}{D}} (\mathbb{E}_{\mathcal{X}_n} (\mathcal{H}^{D-1}(\text{Sk}_n \cap [0, 1]^D)) + 2D). \quad (11)$$

Using the invariance of \mathcal{X}_n by scaling transformations and translations, we observe that

$$\begin{aligned} \mathbb{E}_{\mathcal{X}_n} (\mathcal{H}^{D-1}(\text{Sk}_n \cap [0, 1]^D)) &= \lambda^{-\frac{n(D-1)\beta}{D}} \mathbb{E}_{\mathcal{X}_n} (\mathcal{H}^{D-1}(\text{Sk}_0 \cap [0, \lambda^{\frac{n\beta}{D}}]^D)) \\ &= \lambda^{-\frac{n(D-1)\beta}{D}} \mathbb{E} (\mathcal{H}^{D-1}(\text{Sk}_0 \cap [0, 1]^D)) (\lambda^{\frac{n\beta}{D}})^D. \end{aligned} \quad (12)$$

Combining (10), (11) and (12), we obtain the required result (i).

(ii) The point (i) implies that

$$\mathbb{E}(\text{Vol}([0, 1]^D \setminus \mathcal{O}_{n,n})) = \int_{[0, 1]^D} \mathbb{P}(x \notin \mathcal{O}_{n,n}) dx = \mathbb{P}(0 \notin \mathcal{O}_{n,n}) = O(\lambda^{\frac{n(\beta-H)}{D}}).$$

Therefore, we obtain

$$\mathbb{E}(\text{Vol}([0, 1]^D \setminus W_N)) \leq \sum_{n=N}^{\infty} \mathbb{E}(\text{Vol}([0, 1]^D \setminus \mathcal{O}_{n,n})) \leq \sum_{n=N}^{\infty} O(\lambda^{\frac{n(\beta-H)}{D}}) = O(\lambda^{\frac{N(\beta-H)}{D}}).$$

Finally, using Markov's inequality,

$$\mathbb{P}(\text{Vol}(W_N) < 1/2) = \mathbb{P}(\text{Vol}([0, 1]^D \setminus W_N) \geq 1/2) \leq 2\mathbb{E}(\text{Vol}([0, 1]^D \setminus W_N)) \leq O(\lambda^{\frac{N(\beta-H)}{D}})$$

and the result (ii) follows. \square

1.3. Distribution of the random variable $Z_n(x, y)$.

This subsection is devoted to the calculation of the distribution of $Z_n(x, y)$ conditionally on $\{x \in \mathcal{O}_{n,n}\}$ when $\|x - y\| \leq \tau_n$. In particular, we obtain in Proposition 1.2 below an explicit formula and an upper-bound for the conditional density of $Z_n(x, y)$. A similar method provides in Proposition 1.3 the integrability of the local Lipschitz constant of Z_0 . All these results will play a major role in the estimation of the oscillations of F (see Proposition 1.4) and the application of the Frostman criterion (see Proposition 1.5).

For any $x \in [0, 1]^D$, let $c_n(x)$ (resp. $\mathcal{C}_n(x)$) be the nucleus (resp. the cell) from the Voronoi tessellation \mathcal{T}_n associated with x . Let $c'_n(x)$ be the 'secondary nucleus' of x , i.e. the point of \mathcal{X}_n which is the nucleus of the neighboring cell of $\mathcal{C}_n(x)$ in the direction of the half-line $[c_n(x), x)$. Moreover, for any $z_1 \neq z_2 \in \mathbb{R}^D$ and $x \in \mathbb{R}^D \setminus \{z_1, z_2\}$, we consider

- the bisecting hyperplane H_{z_1, z_2} of $[z_1, z_2]$,
- the cone $\Lambda(z_1, x)$ centered at z_1 and generated by the ball $B_{\tau_n}(x)$,

- the set $A_{n,x}$ of couples (z_1, z_2) with $z_1 \notin B_{\tau_n}(x)$ such that x is between the hyperplane orthogonal to $z_2 - z_1$ and containing z_1 and the parallel hyperplane which is at distance τ_n from H_{z_1, z_2} on the z_1 -side:

$$A_{n,x} = \left\{ (z_1, z_2) \in B_{\tau_n}(x)^c \times \mathbb{R}^D : 0 \leq \langle x - z_1, z_2 - z_1 \rangle \leq \frac{1}{2} \|z_2 - z_1\|^2 \left(1 - \frac{2\tau_n}{\|z_2 - z_1\|}\right) \right\}.$$

Finally, we denote by $\mathcal{V}_n(x, z_1, z_2)$ the volume of the Voronoi flower associated with the intersection $H_{z_1, z_2} \cap \Lambda(z_1, x)$:

$$\mathcal{V}_n(x, z_1, z_2) = \text{Vol} \left(\bigcup \{ B_{\|u - z_1\|}(u) : u \in H_{z_1, z_2} \cap \Lambda(z_1, x) \} \right).$$

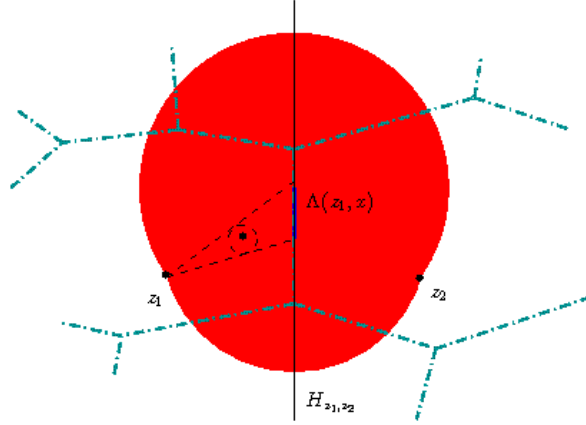


FIGURE 5. The configuration of (z_1, z_2) with the associated Voronoi flower (in red).

Proposition 1.2. *Let $n \geq 0$ and $x, y \in [0, 1]^D$ such that $x \in \mathcal{O}_{n,n}$ and $0 < \|x - y\| \leq \tau_n$. Then*

- (i) *The increment $Z_n(x, y)$ is given by*

$$Z_n(x, y) = -\frac{2\lambda^{-\frac{n\alpha}{D}}}{\|c'_n(x) - c_n(x)\|^2} \langle x - y, c'_n(x) - c_n(x) \rangle. \quad (13)$$

- (ii) *The density z_n of $Z_n(x, y)$ conditionally on $\{x \in \mathcal{O}_{n,n}\}$ is given by (17) for $D \geq 2$ and by (16) for $D = 1$. Moreover, it satisfies*

$$\sup_{t \in \mathbb{R}} z_n(t) \leq \frac{C}{\mathbb{P}(x \in \mathcal{O}_{n,n})} \|x - y\|^{-1} \lambda^{-\frac{n(\beta - \alpha)}{D}} \quad (14)$$

where C is a positive constant depending only on dimension D .

Proof.

- (i) If $x \in \mathcal{O}_{n,n}$ and $\|x - y\| \leq \tau_n$ then

$$\Delta_n(x) = \frac{\text{dist}(x, H_{c_n(x), c'_n(x)})}{\text{dist}(c_n(x), H_{c_n(x), c'_n(x)})}$$

Moreover,

$$\text{dist}(x, H_{c_n(x), c'_n(x)}) = \left\langle x - \frac{c_n(x) + c'_n(x)}{2}, \frac{c_n(x) - c'_n(x)}{\|c_n(x) - c'_n(x)\|} \right\rangle$$

Using (13), we can rewrite the quantity $Z_n(x, y)$ in function of $L_n(x)$ and $u_n(x)$ as

$$Z_n(x, y) = -\frac{2\lambda^{-\frac{n\alpha}{D}}}{L_n(x)} \langle x - y, u_n(x) \rangle.$$

The density of the distribution of $Z_n(x, y)$ conditionally on $\{x \in \mathcal{O}_{n,n}\}$ can then be calculated in the following way: for any non-negative measurable function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$,

$$\mathbb{E}(\psi(Z_n(x, y))) = \frac{\lambda^{2n\beta}}{\mathbb{P}(x \in \mathcal{O}_{n,n})} \int_{B_{\tau_n}(x)^c} J_n(x, y, z_1, \rho, u) dz_1 \quad (15)$$

where $J_n(x, y, z_1, \rho, u)$ is equal to

$$\iint \psi(-2\lambda^{-\frac{n\alpha}{D}} \rho^{-1} \langle x - y, u \rangle) e^{-\lambda^{n\beta} \mathcal{V}'_n(\rho u)} \mathbb{1}_{\mathbb{R}_+}(\langle x - z_1, u \rangle) \rho^{D-1} d\rho d\sigma_{D-1}(u),$$

the domain of integration for ρ being $[2\tau_n + 2\langle x - z_1, u \rangle, \infty)$.

Case $D = 1$. We observe that $u = \pm 1$ and $\mathcal{V}'_n(\rho u) = \rho$. If $u = 1$ (resp. $u = -1$), the range of z_1 is $(-\infty, x - \tau_n)$ (resp. $(x + \tau_n, \infty)$) while for fixed z_1 the range of ρ is $(2\tau_n + 2(x - z_1), \infty)$ (resp. $(2\tau_n + 2(z_1 - x), \infty)$). Consequently, we have

$$\mathbb{E}(\psi(Z_n(x, y))) = \frac{\lambda^{2n\beta}}{\mathbb{P}(x \in \mathcal{O}_{n,n})} \int_{|\rho| > 4\tau_n} \psi(2\lambda^{-n\alpha} \rho^{-1}(x - y)) e^{-\lambda^{n\beta} |\rho|} \left(\frac{|\rho|}{2} - 2\tau_n \right) d\rho.$$

Applying the change of variables $t = 2\lambda^{-n\alpha} \rho^{-1}(x - y)$, we get that the density of $Z_n(x, y)$ is, for $|t| < \lambda^{-n\alpha} \frac{|x-y|}{2\tau_n}$,

$$z_n(t) = \frac{2\lambda^{2n\beta}}{\mathbb{P}(x \in \mathcal{O}_{n,n})} e^{-2\lambda^{n(\beta-\alpha)} \frac{|x-y|}{|t|}} \left(\lambda^{-n\alpha} \frac{|x-y|}{|t|} - 2\tau_n \right) \lambda^{-n\alpha} \frac{|x-y|}{t^2}. \quad (16)$$

In particular,

$$z_n(t) \leq \frac{2\lambda^{2n\beta}}{\mathbb{P}(x \in \mathcal{O}_{n,n})} \frac{\lambda^{-3n\beta+n\alpha}}{|x-y|} \sup_{u>0} (e^{-2u} u^3),$$

which shows (14).

Case $D \geq 2$. We go back to (15). For almost any $u \in \mathbb{S}^{D-1}$, there exists a unique $v \in \mathbb{S}^{D-1} \cap \{y - x\}^\perp$ and a unique $s = \langle u, \frac{y-x}{\|y-x\|} \rangle \in (-1, 1)$ such that

$$u = u_{s,v} = s \frac{y-x}{\|y-x\|} + \sqrt{1-s^2} v.$$

In particular, we can rewrite the uniform measure of \mathbb{S}^{D-1} as

$$d\sigma_{D-1}(u) = (1-s^2)^{\frac{D-3}{2}} ds d\sigma_{D-2}(v).$$

We thus get that $J_n(x, y, z_1, \rho, u)$ is also equal to

$$\iiint \psi(2\lambda^{-\frac{n\alpha}{D}} \|x - y\| s \rho^{-1}) e^{-\lambda^{n\beta} \mathcal{V}'_n(\rho u_{s,v})} \mathbb{1}_{\mathbb{R}_+}(\langle x - y, u_{s,v} \rangle) \rho^{D-1} (1-s^2)^{\frac{D-3}{2}} d\rho ds d\sigma_{D-2}(v),$$

the domain of integration for ρ being $[2\tau_n + 2\langle x - z_1, u_{s,v} \rangle, \infty)$.

We now proceed with the change of variables $\rho = 2\lambda^{-\frac{n\alpha}{D}} \|x - y\| st^{-1}$ with $st > 0$. We then deduce that the density $z_n(t)$ at point t of $Z_n(x, y)$ conditionally on $\{x \in \mathcal{O}_{n,n}\}$ is given by

$$z_n(t) = \frac{\lambda^{2n\beta}}{\mathbb{P}(x \in \mathcal{O}_{n,n})} \iiint J'_n(x, y, z_1, t, s, v) \mathbb{1}_{D_n}(x, y, z_1, t, s, v) ds d\sigma_{D-2}(v) dz_1 \quad (17)$$

where

$$J'_n(x, y, z_1, t, s, v) = e^{-\lambda^{n\beta} \mathcal{V}'_n\left(\frac{2\lambda^{-\frac{n\alpha}{D}} \|x-y\|_s u_{s,v}}{t}\right)} \left(\frac{2\lambda^{-\frac{n\alpha}{D}} \|x-y\|_s}{t}\right)^D \frac{(1-s^2)^{\frac{D-3}{2}}}{t}$$

and

$$D_n = \left\{ (x, y, z_1, t, s, v) : \|x - z_1\| > \tau_n \text{ and } 0 \leq \langle x - z_1, u_{s,v} \rangle \leq \frac{\lambda^{-\frac{n\alpha}{D}} \|x-y\|_s}{t} - \tau_n \right\}.$$

In the sequel, we only deal with the case $t > 0$ but the same could be done likewise for $t < 0$. We denote by x' the intersection of the half-line $[z_1, x)$ with the boundary of the Voronoi cell of z_1 . Moreover, we write $z_1 = x - \gamma w$ where $\gamma > \tau_n$ and $w \in \mathbb{S}^{D-1}$. In particular, we notice that

$$\|x' - z_1\| = \frac{\rho}{2\langle w, u_{s,v} \rangle} = \frac{\lambda^{-\frac{n\alpha}{D}} \|x-y\|_s}{t\langle w, u_{s,v} \rangle}.$$

We can now easily estimate the volume $\mathcal{V}'_n(\cdot)$ in the following way:

$$\mathcal{V}'_n(\cdot) \geq \text{Vol}(B_{\|x'-z_1\|}(x')) = \kappa_D \left(\frac{\lambda^{-\frac{n\alpha}{D}} \|x-y\|_s}{t\langle w, u_{s,v} \rangle} \right)^D. \quad (18)$$

We then proceed with the following change of variables: for almost any $w \in \mathbb{S}^{D-1}$, there exist a unique $\xi = \langle w, u_{s,v} \rangle \in [0, 1)$ and a unique $\eta \in \mathbb{S}^{D-1} \cap \{u_{s,v}\}^\perp$ such that $w = \xi u_{s,v} + \sqrt{1-\xi^2} \eta$ and

$$dz_1 = \gamma^{D-1} d\gamma d\sigma_{D-1}(w) = \gamma^{D-1} (1-\xi^2)^{\frac{D-3}{2}} d\gamma d\xi d\sigma_{D-2}(\eta). \quad (19)$$

In particular, when $(x, y, z_1, t, s, v) \in D_n$, we have

$$0 \leq \gamma = \frac{\langle x - z_1, u_{s,v} \rangle}{\xi} \leq \frac{\lambda^{-\frac{n\alpha}{D}} \|x-y\|_s}{t\xi}. \quad (20)$$

Consequently, for fixed $s, \xi \in (0, 1)$, we have

$$\iiint \mathbb{1}_{D_n} d\gamma d\sigma_{D-2}(v) d\sigma_{D-2}(\eta) \leq \frac{\omega_{D-2}^2}{D} \left(\frac{\lambda^{-\frac{n\alpha}{D}} \|x-y\|_s}{t\xi} \right)^D \quad (21)$$

We deduce from (17), (18), (19) and (21) that the density $z_n(t)$ satisfies, for every $t > 0$,

$$z_n(t) \leq \frac{\lambda^{2n\beta} \omega_{D-2}^2}{D \mathbb{P}(x \in \mathcal{O}_{n,n})} \int_0^1 \int_0^1 J''_n(x, y, t, s, \xi) ds d\xi \quad (22)$$

where

$$J''_n(x, y, t, s, \xi) = e^{-\lambda^{n\beta} \kappa_D \left(\frac{\lambda^{-\frac{n\alpha}{D}} \|x-y\|_s}{t\xi} \right)^D} \frac{(\sqrt{2} \lambda^{-\frac{n\alpha}{D}} \|x-y\|_s)^{2D}}{t^{2D+1} \xi^D} ((1-s^2)(1-\xi^2))^{\frac{D-3}{2}}.$$

We then use the change of variables $\tau = \lambda^{-\frac{n\alpha}{D}} \|x-y\|_s (t\xi)^{-1}$.

Subcase $D \geq 3$. Using that $(1 - s^2)(1 - \xi^2)$ and ξ are bounded by 1, we obtain from the change of variables that

$$\begin{aligned} z_n(t) &\leq \frac{\lambda^{2n\beta} \omega_{D-2}^2}{D\mathbb{P}(x \in \mathcal{O}_{n,n})} \int_0^1 \int_{\{\tau>0\}} \tau^D e^{-\lambda^{n\beta} \kappa_D \tau^D} \frac{(2\tau\xi)^D}{t} \frac{t\xi}{\lambda^{-\frac{n\alpha}{D}} \|x-y\|} d\tau d\xi \\ &\leq \frac{C}{\mathbb{P}(x \in \mathcal{O}_{n,n})} \frac{\lambda^{2n\beta + \frac{n\alpha}{D}}}{\|x-y\|} \int_{\{\tau>0\}} \tau^{2D} e^{-\lambda^{n\beta} \kappa_D \tau^D} d\tau \\ &= \frac{C'}{\mathbb{P}(x \in \mathcal{O}_{n,n})} \frac{\lambda^{-\frac{n(\beta-\alpha)}{D}}}{\|x-y\|} \end{aligned}$$

where C and C' are two positive constants which only depend on D .

Subcase $D = 2$. We return to (22) and apply the same change of variables $\tau = \lambda^{-\frac{n\alpha}{D}} \|x-y\| s(t\xi)^{-1}$. We now obtain that

$$z_n(t) \leq \frac{8\lambda^{2n\beta} \lambda^{\frac{n\alpha}{2}}}{\|x-y\| \mathbb{P}(x \in \mathcal{O}_{n,n})} \int_{\{\tau>0\}} \tau^4 e^{-\lambda^{n\beta} \pi \tau^2} (\dots) d\tau \quad (23)$$

where

$$(\dots) = \int_{\xi=0}^{1 \wedge \frac{\lambda^{-\frac{n\alpha}{2}} \|x-y\|}{t\tau}} (1-\xi)^{-\frac{1}{2}} \left(1 - \frac{t\xi\tau}{\lambda^{-\frac{n\alpha}{2}} \|x-y\|}\right)^{-\frac{1}{2}} d\xi.$$

We notice that there exists a positive constant $C > 0$ such that for every $a > 0$, we have

$$\int_{\xi=0}^{1 \wedge a} (1-\xi)^{-\frac{1}{2}} \left(1 - \frac{\xi}{a}\right)^{-\frac{1}{2}} d\xi \leq C |\log |a-1||. \quad (24)$$

Indeed, due to the facts that the left-hand side of (24) is bounded for large a and that the calculation is symmetric with respect to 1, it is enough to look for the behaviour of the Abelian-type integral when $a > 1$ is close to 1. A direct calculation shows then that

$$\int_0^1 (1-\xi)^{-\frac{1}{2}} \left(1 - \frac{\xi}{a}\right)^{-\frac{1}{2}} d\xi = \sqrt{a} \operatorname{argch} \left(\frac{a+1}{a-1} \right) \underset{a \rightarrow 1}{\sim} -\log(a-1),$$

which proves (24). Consequently, we get from (23) and (24) that

$$z_n(t) \leq \frac{C' \lambda^{2n\beta} \lambda^{\frac{n\alpha}{2}}}{\|x-y\| \mathbb{P}(x \in \mathcal{O}_{n,n})} \int_{\{\tau>0\}} \tau^4 e^{-\lambda^{n\beta} \pi \tau^2} \left| \log \left| 1 - \frac{\lambda^{-\frac{n\alpha}{2}} \|x-y\|}{t\tau} \right| \right| d\tau,$$

where C' denotes again a positive constant.

We now fix $\varepsilon \in (0, 1)$ and we split the integral:

- on the range of τ which satisfy $|1 - \frac{\lambda^{-\frac{n\alpha}{2}} \|x-y\|}{t\tau}| > \varepsilon$, the upper-bound is similar to the case $D \geq 3$;

- on the range of τ satisfying $|1 - \frac{\lambda^{-\frac{n\alpha}{2}} \|x-y\|}{t\tau}| < \varepsilon$, the integral is bounded by

$$\frac{1}{(1-\varepsilon)^6} e^{-\lambda^{n\beta} \pi \frac{\lambda^{-\frac{n\alpha}{2}} \|x-y\|^2}{(1+\varepsilon)^2 t^2}} \left(\frac{\lambda^{-\frac{n\alpha}{2}} \|x-y\|}{t} \right)^5 \int_{u=1-\varepsilon}^{1+\varepsilon} |\log |1-u^{-1}|| du \leq C' \varphi \left(\frac{\lambda^{-\frac{n\alpha}{2}} \|x-y\|}{t} \right)$$

where $\varphi(u) = e^{-\lambda^{n\beta} \pi \frac{u^2}{(1+\varepsilon)^2}} u^5$, $u > 0$.

It remains to notice that the maximum of the function φ is of order $O(\lambda^{-\frac{5}{2}n\beta})$ to deduce the required result (14). \square

We conclude this subsection with the integrability of the Lipschitz constant $L(x)$ of the affine part of Δ_0 above x , i.e.

$$L(x) = \frac{2}{\|c_0(x) - c'_0(x)\|}. \quad (25)$$

Proposition 1.3. *For every $x \in [0, 1]^D$, $\mathbb{E}(L(x)) < \infty$ and $\sup_{n, N \in \mathbb{N}} \mathbb{E}(L(x) | x \in \mathcal{O}_{n, N}) < \infty$.*

Proof. We start by noticing that the conditioning can be easily overlooked. Indeed, the condition $\{0 \in \mathcal{O}_{n, N}\}$ implies that a vicinity of the origin is in the same simplex of \mathcal{S}_0 so $c_0(0)$ and $c'_0(0)$ are farther. Consequently, we have $\mathbb{E}(L(x) | x \in \mathcal{O}_{n, N}) \leq \mathbb{E}(L(x))$ for every $n, N \geq 0$.

For $D = 1$, the integrability of the variable $L(x)$ given by (25) comes from the fact that the distance from the two neighbors of x (the nearest and second nearest) is Gamma-distributed. When $D \geq 2$, we use a reasoning similar to the proof of Proposition 1.2 to obtain that

$$\begin{aligned} \mathbb{E}(L(x)) &= \mathbb{E}\left(\sum_{z_1 \neq z_2} \frac{2}{\|z_1 - z_2\|} \mathbb{I}_{(c_n(x), c'_n(x))}(z_1, z_2)\right) \\ &= \iint \frac{2}{\|z_1 - z_2\|} \mathbb{P}((x + \mathbb{R}_+(x - z_1)) \cap H_{z_1, z_2} \in \mathcal{C}_{z_1} \cap \mathcal{C}_{z_2}) dz_1 dz_2. \end{aligned} \quad (26)$$

We write $z_2 = z_1 + \rho u$ where $u \in \mathbb{S}^{D-1}$, $\rho > 0$ and $u = s \frac{x - z_1}{\|x - z_1\|} + \sqrt{1 - s^2} v$ where $s \in (0, 1)$ and $v \in \mathbb{S}^{D-1} \cap \{x - z_1\}^\perp$. In particular, the distance from z_1 to the point $(x + \mathbb{R}_+(x - z_1)) \cap H_{z_1, z_2}$ is $\frac{\rho}{2s}$. Consequently, we deduce from a change of variables applied to the integral in (26) that

$$\mathbb{E}(L(x)) = \iint \int_{s=0}^1 \frac{2\omega_{D-2}}{\rho} \mathbb{I}_{\mathbb{R}_+}(\rho - 2\|x\|s) e^{-\kappa_D(\frac{\rho}{2s})^D} \rho^{D-1} (1 - s^2)^{\frac{D-3}{2}} ds d\rho dx.$$

When $D \geq 3$, we proceed with the change of variables $\tau = \frac{\rho}{2s}$. There is a constant $C > 0$ such that

$$\begin{aligned} \mathbb{E}(L(x)) &\leq \omega_{D-2} \int_{\rho=0}^{\infty} \rho^{D-1} \int_{\tau=\frac{\rho}{2}}^{\infty} \left(\int_{B(0, \tau)} dx \right) e^{-\kappa_D u^D} \frac{d\tau}{\tau^2} d\rho \\ &\leq C \int_{\rho=0}^{\infty} \rho^{D-1} \int_{\tau=\frac{\rho}{2}}^{\infty} \tau^{D-2} e^{-\kappa_D \tau^D} d\tau d\rho \\ &\leq C \int_{\rho=0}^{\infty} \rho^{D-1} e^{-\frac{\kappa_D}{2}(\frac{\rho}{2})^D} d\rho \int_{\tau=0}^{\infty} \tau^{D-2} e^{-\frac{\kappa_D}{2}\tau^D} d\tau < \infty. \end{aligned}$$

Finally, when $D = 2$, with the same change of variables, we get

$$\mathbb{E}(L) \leq C \int_{\rho=0}^{\infty} \rho \int_{\tau=\frac{\rho}{2}}^{\infty} e^{-\pi\tau^2} \frac{d\tau d\rho}{\sqrt{1 - (\frac{\rho}{2\tau})^2}}. \quad (27)$$

We treat separately the integral in τ for fixed $\rho > 0$:

$$\begin{aligned} \int_{\tau=\frac{\rho}{2}}^{\infty} e^{-\pi\tau^2} \frac{d\tau}{\sqrt{1 - (\frac{\rho}{2\tau})^2}} &\leq 2\rho e^{-\pi(\frac{\rho}{2})^2} \int_{\tau=\frac{\rho}{2}}^{\rho} \frac{\rho}{2\tau^2} \frac{d\tau}{\sqrt{1 - (\frac{\rho}{2\tau})^2}} + \sqrt{\frac{4}{3}} \int_{\tau=\rho}^{\infty} e^{-\pi\tau^2} d\tau \\ &\leq 2\rho e^{-\pi(\frac{\rho}{2})^2} \left[\arccos\left(\frac{\rho}{2u}\right) \right]_{\frac{\rho}{2}}^{\rho} + \sqrt{\frac{4}{3}} e^{-\frac{\pi}{2}\rho^2} \int_{\tau=0}^{\infty} e^{-\frac{\pi}{2}\tau^2} d\tau \\ &\leq \frac{2\pi}{3} \rho e^{-\pi(\frac{\rho}{2})^2} + C' e^{-\frac{\pi}{2}\rho^2}, \end{aligned} \quad (28)$$

where C' is a positive constant. Inserting (28) in (27), we get the required result. \square

1.4. Size of the increments of F .

The box-dimension of Γ , as well as its Hausdorff dimension, is closely related to the oscillations of F (see [13, 14]). Let us recall that, for every $A \subset [0, 1]^D$, the oscillation of F over A is defined by

$$\text{osc}(A) = \sup_{y, y' \in A} |F(y') - F(y)|. \quad (29)$$

In particular, we will consider, for all $x \in [0, 1]^D$ and $\tau > 0$, the oscillation of F over the cube $x + [0, \tau]^D$ given by

$$\text{osc}_\tau(x) = \text{osc}(x + [0, \tau]^D) = \sup_{y, y' \in x + [0, \tau]^D} |F(y') - F(y)|. \quad (30)$$

Proposition 1.4. *Let $0 < p < \frac{\alpha}{H}$. Then, for every $x \in [0, 1]^D$, we have, when $N \rightarrow \infty$,*

$$\mathbb{E}(\text{osc}_{\tau_N}(x)) = \mathbb{E}(\text{osc}_{\tau_N}(0)) = O(\tau_N^p).$$

Proof. Let us write

$$\delta_N = \sup_{x \in [0, \tau_N]^D} |F(x) - F(0)|.$$

We claim that

$$\mathbb{P}(\liminf \{\delta_N \leq \tau_N^p\}) = 1. \quad (31)$$

By Markov's inequality,

$$\mathbb{P}(\delta_N \geq \tau_N^p) \leq \tau_N^{-p} \mathbb{E}(\delta_N). \quad (32)$$

We can write

$$\mathbb{E}(\delta_N) \leq \mathbb{E}\left(\sum_{n=0}^{\infty} \sup_{x \in [0, \tau_N]^D} |Z_n(x, 0)|\right) = S_1(N) + S_2(N) + S_3(N)$$

with

$$\begin{aligned} S_1(N) &= \sum_{n=0}^N \mathbb{E}\left(\sup_{x \in [0, \tau_N]^D} |Z_n(x, 0)| \mid 0 \in \mathcal{O}_{n,N}\right) \mathbb{P}(0 \in \mathcal{O}_{n,N}), \\ S_2(N) &= \sum_{n=0}^N \mathbb{E}\left(\sup_{x \in [0, \tau_N]^D} |Z_n(x, 0)| \mid 0 \notin \mathcal{O}_{n,N}\right) \mathbb{P}(0 \notin \mathcal{O}_{n,N}), \\ S_3(N) &= \sum_{n=N+1}^{\infty} \mathbb{E}\left(\sup_{x \in [0, \tau_N]^D} |Z_n(x, 0)|\right). \end{aligned}$$

Since $[0, \tau_N]^D \subset B_{\sqrt{D}\tau_N}(0)$, we mention that for the purpose of this proof, the definition (6) of the set $\mathcal{O}_{n,N}$ should be slightly adapted by substituting $\sqrt{D}\tau_N$ for τ_N . For sake of simplicity, we omit that technical detail.

For $S_1(N)$, we notice that

$$\mathbb{E}\left(\sup_{x \in [0, \tau_N]^D} |Z_n(x, 0)| \mid 0 \in \mathcal{O}_{n,N}\right) \leq (\lambda^{-\frac{n\alpha}{D}} \sqrt{D} \tau_N) (\lambda^{\frac{n\beta}{D}} \mathbb{E}(L(0) \mid 0 \in \mathcal{O}_{n,N}))$$

where $L(0)$ is the Lipschitz constant of Δ_0 at 0 when the underlying Poisson point process is homogeneous of intensity 1. Thanks to Proposition 1.3, we have $\sup_{n,N \in \mathbb{N}} \mathbb{E}(L(0) \mid 0 \in \mathcal{O}_{n,N}) < \infty$.

Thus, using $0 < \alpha \leq \beta \leq 1$, we obtain

$$S_1(N) \leq \sum_{n=0}^N \lambda^{-\frac{n\alpha}{D}} \lambda^{-\frac{NH}{D}} \lambda^{\frac{n\beta}{D}} \sqrt{D} \sup_{n, N \in \mathbb{N}} \mathbb{E}(L(0) | 0 \in \mathcal{O}_{n,N}) \leq \begin{cases} C_1 \lambda^{\frac{(\beta-\alpha-H)N}{D}} & \text{if } \alpha < \beta \\ C'_1 N \lambda^{-\frac{NH}{D}} & \text{if } \alpha = \beta \end{cases}$$

where C_1, C'_1 are two positive constants which do not depend on N .

For $S_2(N)$ we use the upper estimate (see (4))

$$\mathbb{E} \left(\sup_{x \in [0, \tau_N]^D} |Z_n(x, 0)| \mid 0 \notin \mathcal{O}_{n,N} \right) \leq \lambda^{-\frac{n\alpha}{D}}$$

and Proposition 1.1(i) to get

$$S_2(N) \leq \sum_{n=0}^N \lambda^{-\frac{n\alpha}{D}} \lambda^{\frac{n\beta-NH}{D}} \leq \begin{cases} C_2 \lambda^{\frac{(\beta-\alpha-H)N}{D}} & \text{if } \alpha < \beta \\ C'_2 N \lambda^{-\frac{NH}{D}} & \text{if } \alpha = \beta \end{cases}$$

where C_2, C'_2 are two positive constants which do not depend on N .

Finally, for $S_3(N)$ we only use the upper estimate

$$\mathbb{E} \left(\sup_{x \in [0, \tau_N]^D} |Z_n(x, 0)| \right) \leq \lambda^{-\frac{n\alpha}{D}}$$

to get

$$S_3(N) \leq \sum_{n=N+1}^{\infty} \lambda^{-\frac{n\alpha}{D}} \leq C_3 \lambda^{-\frac{N\alpha}{D}}$$

where $C_3 > 0$ is a positive constant which does not depend on N .

Therefore,

$$\mathbb{E} \left(\sum_{n=0}^{\infty} \sup_{x \in [0, \tau_N]^D} |Z_n(x, 0)| \right) \leq \begin{cases} C(\tau_N^{1-\frac{\beta-\alpha}{H}} + \tau_N^{\frac{\alpha}{H}}) & \text{if } \alpha < \beta \\ C'(|\log(\tau_N)|\tau_N + \tau_N^{\frac{\alpha}{H}}) & \text{if } \alpha = \beta \end{cases}$$

where C, C' are two positive constants which do not depend on N . The rhs of (32) is in particular summable in N as soon as $p < \frac{\alpha}{H}$ (which guarantees that $1 - \frac{\beta-\alpha}{H} > p$ since $1 - \frac{\beta}{H} > 0$). Consequently, by Borel-Cantelli's lemma, (31) holds. Then, we obtain

$$\text{osc}_{\tau_N}(0) \leq 2\delta_N = O(\tau_N^p)$$

almost surely for all N large enough. To conclude, let us notice that, by stationarity, it follows that

$$\mathbb{E}(\text{osc}_{\tau_N}(x)) = \mathbb{E}(\text{osc}_{\tau_N}(0)) = O(\tau_N^p).$$

for all $x \in [0, 1]^D$. □

We conclude this subsection with an estimate of the expectation of a particular functional of the increment $F(x) - F(y)$ that will appear in the application of a Frostman-type lemma in the next section.

Proposition 1.5. *Let $s > 1$ and $n \geq 0$. If $x, y \in [0, 1]^D$ satisfy $\tau_{n+1} < \|x - y\| \leq \tau_n$ then*

$$\mathbb{E} \left((|F(x) - F(y)|^2 + \|x - y\|^2)^{-\frac{s}{2}} \mathbb{I}_{\mathcal{O}_{n,n}}(x) \right) \leq C \|x - y\|^{-s + \frac{\beta-\alpha}{H}} \quad (33)$$

where $C > 0$ is a constant which do not depend on x, y nor on n .

Proof. Remember that $F(x) - F(y) = Z_n(x, y) + S_n(x, y)$ where $Z_n(x, y)$ and $S_n(x, y)$ are independent. Let \mathbb{P}_n be the probability associated with \mathcal{X}_n and μ_{S_n} be the probability distribution of the random variable $S_n(x, y)$. From Proposition 1.2 one obtains

$$\begin{aligned}
 & \mathbb{E} \left((|F(x) - F(y)|^2 + \|x - y\|^2)^{-\frac{s}{2}} \mathbb{1}_{\mathcal{O}_{n,n}}(x) \right) \\
 &= \iint ((Z_n(x, y) + v)^2 + \|x - y\|^2)^{-\frac{s}{2}} \mathbb{1}_{\mathcal{O}_{n,n}}(x) d\mathbb{P}_n d\mu_{S_n}(v) \\
 &= \iint \mathbb{P}(x \in \mathcal{O}_{n,n}) ((u + v)^2 + \|x - y\|^2)^{-\frac{s}{2}} z_n(u) du d\mu_{S_n}(v) \\
 &\leq \int \mathbb{P}(x \in \mathcal{O}_{n,n}) \left(\int_{\{|u+v| < \|x-y\|\}} \|x - y\|^{-s} z_n(u) du d\mu_{S_n}(v) + \int_{\{|u+v| > \|x-y\|\}} |u + v|^{-s} z_n(u) du d\mu_{S_n}(v) \right) \\
 &\leq 2\|x - y\| \sup_{t \in \mathbb{R}} (z_n(t)) \|x - y\|^{-s} + \sup_{t \in \mathbb{R}} (z_n(t)) \int_{\{|u+v| > \|x-y\|\}} |u + v|^{-s} du d\mu_{S_n}(v) \\
 &\leq 2C\|x - y\| \|x - y\|^{-1} \lambda^{-\frac{n(\beta-\alpha)}{D}} \|x - y\|^{-s} + C\|x - y\|^{-1} \lambda^{-\frac{n(\beta-\alpha)}{D}} \|x - y\|^{-s+1} \\
 &= C\|x - y\|^{-s} \lambda^{-\frac{n(\beta-\alpha)}{D}}.
 \end{aligned}$$

Finally, the assumption on $\|x - y\|$ implies that $\lambda^{-n} \leq (\|x - y\| \lambda^{\frac{H}{D}})^{\frac{D}{H}}$, which provides the desired bound. \square

2. PROOF OF THE MAIN THEOREM

Let us recall that for any non-empty compact set K one has $0 \leq \dim_H(K) \leq \dim_B(K)$ (see [14]). Thus the proof of Theorem 1 will consist in proving that $D + 1 - \frac{\alpha}{\beta}$ is an upper bound for $\dim_B(\Gamma)$ and a lower bound for $\dim_H(\Gamma)$.

2.1. An upper bound for the box-dimension of Γ .

First we investigate the box-dimension of Γ . For every $\tau > 0$ we cover $[0, 1]^D \times \mathbb{R}$ with τ -mesh cubes and denote by $\mathcal{N}(\tau)$ the (finite) number of cubes from this partition which intersect Γ . Then, we can express the box-dimension of Γ in terms of $\mathcal{N}(\tau_N)$ (see Section 3.1 in [14] and Section 2.2 in [36]):

$$\dim_B(\Gamma) = \limsup_{N \rightarrow \infty} \frac{\log \mathcal{N}(\tau_N)}{-\log(\tau_N)}. \quad (34)$$

Lemma 2.1. *There exists a constant $d > 0$ such that $\mathbb{P}(\dim_B(\Gamma) = d) = 1$.*

Proof. For every $m \in \mathbb{N}$, let us denote by \mathcal{A}_m the σ -algebra generated by the point process \mathcal{X}_m . We observe that the graph of the function which associates to any $x \in [0, 1]^D$ the quantity $F(x) - \sum_{n=0}^m \lambda^{-\frac{n\alpha}{D}} \Delta_n(x)$ has the same box-dimension as the graph of F . Consequently, we can use (34) to show that $\dim_B(\Gamma)$ is a random variable which is measurable with respect to $\sigma(\mathcal{A}_m : m \geq n)$ for every $m \in \mathbb{N}$. We then use the 0-1 law to deduce that it is almost surely constant. \square

Proof of the upper bound of (3).

Using (34) as if $\limsup_{N \rightarrow \infty}$ is a real limit, provided that τ_N is replaced with a subsequence, applying first Lebesgue's convergence theorem to $\frac{\log \mathcal{N}(\tau_N)}{-\log(\tau_N)} \leq D + 1$ and finally Jensen's

inequality to the concave function \log , we obtain

$$\mathbb{E}(\dim_B(\Gamma_F)) = \lim_{N \rightarrow \infty} \mathbb{E} \left(\frac{\log \mathcal{N}(\tau_N)}{-\log(\tau_N)} \right) \leq \lim_{N \rightarrow \infty} \log \mathbb{E} \left(\frac{\mathcal{N}(\tau_N)}{-\log(\tau_N)} \right).$$

We now use Proposition 11.1 from [14] and Proposition 1.4 to get

$$\mathbb{E}(\mathcal{N}(\tau_N)) \leq 2 \lceil \tau_N^{-1} \rceil^D + \tau_N^{-1} \sum_{\substack{\mathbf{k}=(k_1, \dots, k_D) \\ 0 \leq k_i \leq \lceil \tau_N^{-1} \rceil}} \mathbb{E}(\text{osc}_{\tau_N}(\mathbf{k}\tau_N)) = O(\tau_N^{p-D-1}).$$

In conclusion, for every $H > \beta$ and $p < \frac{\alpha}{H}$, we have $\mathbb{E}(\dim_B(\Gamma)) \leq D + 1 - p$. We finally get

$$\dim_B(\Gamma) = \mathbb{E}(\dim_B(\Gamma)) \leq D + 1 - \frac{\alpha}{\beta}$$

by letting $H \rightarrow \beta$, $p \rightarrow \frac{\alpha}{\beta}$, and by using Lemma 2.1. \square

2.2. A lower bound for the Hausdorff dimension of Γ .

To find a lower bound for the Hausdorff dimension of a compact set is generally a difficult problem. An important step was made in [19] when Hunt proposed a way to find a lower bound for the Hausdorff dimension of the graph of Weierstrass functions using the ‘finite energy criterion’. This well-known criterion is used for calculating the Hausdorff dimension of more general fractal sets (see e.g. [29], Chapter 4). However, the arguments of [19] cannot be applied for all Weierstrass-type functions, in particular it rules out the Takagi-Knopp series because the sawtooth function Δ is not regular enough.

Let us recall that the Hausdorff dimension of a non-empty compact set K may be expressed in terms of finite energy of some measures thanks to a lemma due to Frostman (see e.g. [20, 25]):

$$\dim_H(K) = \sup_{\mu} (\sup\{s \geq 0 : I_s(\mu) < \infty\})$$

where the supremum on μ is taken over all the finite and non-null Borel measures such that $\mu(K) > 0$, $I_s(\mu)$ being the s -energy of μ defined by

$$I_s(\mu) = \iint \|x - y\|^{-s} d\mu(x) d\mu(y). \quad (35)$$

Therefore, if such a measure μ satisfies $I_s(\mu) < \infty$ then $\dim_H(K) \geq s$.

Let us recall now a classical way to construct such a measure on Γ . For each $N \geq 0$ the set W_N is a Borel subset of $[0, 1]^D$. Since F is a continuous function then Γ is a Borel set too so that we can consider the measure μ_{W_N} obtained by lifting onto Γ the D -dimensional Lebesgue measure restricted to W_N . Precisely, for all Borel set $E \subset \mathbb{R}^{D+1}$,

$$\mu_{W_N}(E) = \text{Vol}\{x \in [0, 1]^D \cap W_N \text{ such that } (x, F(x)) \in E\}$$

and μ_{W_N} is a positive measure as soon as $\text{Vol}(W_N) > 0$. Using Euclidean norm on $\mathbb{R}^D \times \mathbb{R}$ we can recast (35) into

$$I_s(\mu_{W_N}) = \iint_{W_N \times W_N} (|F(x) - F(y)|^2 + \|x - y\|^2)^{-\frac{s}{2}} dx dy. \quad (36)$$

Notice that the finiteness of $I_s(\mu_{W_N})$ depends only on the size of the increments $F(x) - F(y)$ when $\|x - y\|$ is small.

Proof of the lower bound of (3).

Let us consider, for all $n \geq 0$, the set

$$T_n = \{(x, y) \in [0, 1]^D \times [0, 1]^D : \tau_{n+1} < \|x - y\| \leq \tau_n\}.$$

Let $N \geq 1$ and $s > 1$. Since $W_N \subset \mathcal{O}_{n,n}$ for all $n \geq N$, we have, with (36),

$$\begin{aligned} I_s(\mu_{W_N}) &\leq \iint_{(x,y) \in W_N \times [0,1]^D} (|F(x) - F(y)|^2 + \|x - y\|^2)^{-\frac{s}{2}} dx dy \\ &\leq C_N + \sum_{n=N}^{\infty} \iint_{(x,y) \in T_n \cap (W_N \times [0,1]^D)} (|F(x) - F(y)|^2 + \|x - y\|^2)^{-\frac{s}{2}} dx dy \\ &\leq C_N + \sum_{n=N}^{\infty} \iint_{(x,y) \in T_n} (|F(x) - F(y)|^2 + \|x - y\|^2)^{-\frac{s}{2}} \mathbb{1}_{\mathcal{O}_{n,n}}(x) dx dy \end{aligned}$$

where

$$\begin{aligned} C_N &= \iint_{(x,y) \in (T_0 \cup \dots \cup T_{N-1}) \cap (W_N \times [0,1]^D)} (|F(x) - F(y)|^2 + \|x - y\|^2)^{-\frac{s}{2}} dx dy \\ &\leq \iint_{(x,y) \in (T_0 \cup \dots \cup T_{N-1})} \|x - y\|^{-s} dx dy \leq \lambda^{NHs}. \end{aligned}$$

To show that the integral (36) is finite almost surely it is enough to show that its expectation is finite. By Fubini's theorem,

$$\mathbb{E}(I_s(\mu_{W_N})) \leq \lambda^{NHs} + \sum_{n=N}^{\infty} \iint_{(x,y) \in T_n} \mathbb{E}((|F(x) - F(y)|^2 + \|x - y\|^2)^{-\frac{s}{2}} \mathbb{1}_{\mathcal{O}_{n,n}}(x)) dx dy.$$

Using the majoration (33) we obtain

$$\mathbb{E}(I_s(\mu_{W_N})) \leq \lambda^{NHs} + C \iint_{(x,y) \in \cup_{n=N}^{\infty} T_n} \|x - y\|^{-s + \frac{\beta - \alpha}{H}} dx dy.$$

This latter integral converges as soon as $-s + \frac{\beta - \alpha}{H} > -D$. Therefore the random measure μ_{W_N} has a finite s -energy for all $1 < s < D + \frac{\beta - \alpha}{H}$. Since N may be chosen such that $\text{Vol}(W_N) > 0$ with an arbitrary high probability (see Proposition 1.1) we deduce that

$$\dim_{\text{H}}(\Gamma) \geq D + \frac{\beta - \alpha}{H}$$

for all $H > \beta$ and almost surely. We obtain the desired lower bound by letting $H \in \mathbb{Q}_+$ go to β . \square

3. RELATED MODELS

In this section, we study the fractal properties of two different models which are related to our Poisson-Voronoi construction: a deterministic series of pyramidal functions with hexagonal bases on the one hand, a random perturbation of the classical Takagi-Knopp series on a dyadic mesh on the other hand.

3.1. Takagi-like series directed by hexagonal Voronoi tessellations.

The series that we study here is very close to the original function $F_{\lambda,\alpha,\beta}$. The novelty lies in the Δ_n signals: we consider now pyramids with a regular hexagonal basis. This model is naturally related to the previous one for two reasons: first an hexagonal mesh is classically known to be the Voronoi tessellation generated by a regular triangular mesh. Secondly, the mean of the number of vertices of a typical cell from a Poisson-Voronoi tessellation is known to be 6 (see e.g. [28], Prop. 3.3.1.) so that an hexagonal mesh may be seen as an *idealized* realization of a Poisson-Voronoi tessellation.

We start with the deterministic Voronoi tessellation whose cells are identical regular hexagons such that one is centered at the origin $(0, 0)$ and has a vertex at $(1, 0)$. Then, considering all the centers of these hexagons as a point process \mathcal{X}_0 , we set $\mathcal{X}_n = 2^{-n}\mathcal{X}_0$ and construct the hexagonal Voronoi tessellation associated with. Here again $\Delta_n : \mathbb{R}^2 \rightarrow [0, 1]$ is the pyramidal function satisfying $\Delta_n = 0$ on $\bigcup_{c \in \mathcal{X}_n} \partial \mathcal{C}_c$ and $\Delta_n = 1$ on \mathcal{X}_n .

Let us notice that, for all $x \in \mathbb{R}^2$ and all $n \geq 0$, we have $\Delta_n(x) = \Delta(2^n x)$. We fix $\alpha \in (0, 1]$ and define a function

$$f_\alpha(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} \Delta_n(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} \Delta(2^n x). \quad (37)$$

The main theorem of this section is the analogue of Theorem 1. Actually we state a more precise result than Proposition 1.4 for the oscillation of f_α but we cannot determine the Hausdorff dimension of Γ_α .

Theorem 3.1. *Let $0 < \alpha \leq 1$. Then f_α is a continuous function such that*

$$\exists C, C' > 0, \quad \forall \tau \in (0, 1), \quad \forall x \in [0, 1]^2, \quad C' \tau^\alpha \leq \text{osc}_\tau(x) \leq C \tau^\alpha. \quad (38)$$

Moreover, its random graph

$$\Gamma_\alpha = \{(x, f_\alpha(x)) : x \in [0, 1]^2\} \subset \mathbb{R}^2 \times \mathbb{R}$$

is a fractal set satisfying

$$\dim_B(\Gamma_\alpha) = 3 - \alpha. \quad (39)$$

Proof. In the sequel we drop again the index α so that $f = f_\alpha$ and $\Gamma = \Gamma_\alpha$. We also keep the notation $Z_n(x, y) = 2^{-n\alpha}(\Delta_n(x) - \Delta_n(y)) = 2^{-n\alpha}(\Delta(2^n x) - \Delta(2^n y))$ for all $x, y \in \mathbb{R}^2$.

(i) Let us state the upper estimates first. We fix $x, y \in [0, 1]^2$ such that $\|x - y\| \in (0, 1)$ and consider $N \geq 1$ such that $2^{-N} < \|x - y\| \leq 2^{-(N-1)}$. Using the fact that Δ is Lipschitz, with Lipschitz constant 1, and bounded by 1, we have $|Z_n(x, y)| \leq \min(\|x - y\|, 1)$. Therefore

$$|f(x) - f(y)| \leq \sum_{n=0}^{\infty} |Z_n(x, y)| \leq \sum_{n=0}^{N-1} (2^{n(1-\alpha)} \|x - y\|) + \sum_{n=N}^{\infty} 2^{-n\alpha} \leq C \|x - y\|^\alpha$$

where C is a positive constant which only depends on α . Then, $|f(x) - f(y)| \leq C \tau^\alpha$ for all $x, y \in \mathbb{R}^2$ such that $0 < \|x - y\| < \sqrt{2}\tau < 1$. This gives the upper bound in (38).

(ii) Now we state the lower estimates. We begin with finding a lower bound for the oscillation over an hexagonal cell. The key-point for estimating this oscillation is to calculate two particular increments for two pairs of well-chosen points belonging to the cell, namely the center and vertices. To keep in mind the number of the generation, we denote by \mathcal{C}_c^n , $c \in \mathcal{X}_n$, the cells of the Voronoi tessellation of generation n . Moreover, the six vertices of a cell \mathcal{C}_c^n are denoted by c_i with $i \in \{1, \dots, 6\}$.

Then, let $N \geq 1$, $c \in \mathcal{X}_N$ and $i \in \{1, \dots, 6\}$. We can write

$$f(c) - f(c_i) = \sum_{n=0}^{N-1} Z_n(c, c_i) + \sum_{n=N}^{\infty} Z_n(c, c_i).$$

As soon as a point is the center (resp. a vertex) of a cell \mathcal{C}_c^N then it is the center (resp. a vertex) of all the cells of higher generations. Hence $\Delta(2^n c) = 1$ and $\Delta(2^n c_i) = 0$ for all $n \geq N$. Therefore

$$\sum_{n=N}^{\infty} Z_n(c, c_i) = \sum_{n=N}^{\infty} 2^{-n\alpha} = \frac{1}{1-2^{-\alpha}} 2^{-N\alpha}.$$

Let $i, j \in \{1, \dots, 6\}$. We obtain

$$\begin{aligned} \sup_{y, y' \in \mathcal{C}_c^N} |f(y') - f(y)| &\geq \frac{1}{2} ((f(c) - f(c_i)) + (f(c) - f(c_j))) \\ &\geq \frac{1}{2} \left(\sum_{n=0}^{N-1} Z_n(c, c_i) + \sum_{n=0}^{N-1} Z_n(c, c_j) \right) + \frac{2}{1-2^{-\alpha}} 2^{-N\alpha}. \end{aligned}$$

For a fixed hexagon \mathcal{C}_c^N , we claim that there exists a pair of two diametrically opposite vertices c_i and c_j such that the sum of the two first sums above is positive. Indeed, for $n < N$, the center c is included in one or two cells \mathcal{C}_c^n of the tessellation of generation n and it can be only in three positions: at the center c' , on one 'edge' (i.e. a segment between two consecutive vertices c'_i and c'_{i+1} or between the center c' and its vertex c'_i), or on a 'face' (i.e. an open triangle of vertices c' , c'_i and c'_{i+1}). Let us denote by C , E and F respectively these three positions. The sequence of positions when n goes from $N-1$ to 0 has to be (C, \dots, C) or $(C, \dots, C, E, \dots, E)$ or $(C, C, \dots, C, E, \dots, E, F, \dots, F)$ (see the first generations on Figure 7 below).

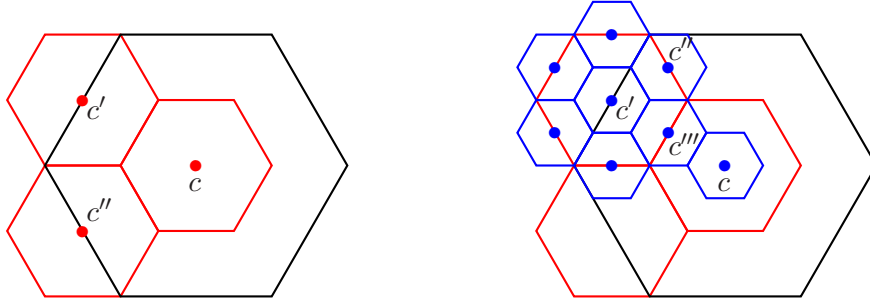


FIGURE 7. The only three possibilities for the sequence of the positions of the centers of the cells. On the left p goes from 1 to 0: c gives (C, C) , c' gives (C, E) and c'' gives (C, E) . On the right p goes from 2 to 0: c gives (C, C, C) , c' gives (C, C, E) , c'' gives (C, E, E) and c''' gives (C, E, F) .

In the first case, any pair of vertices will make the sum of the two sums above non-negative. In the second and third cases, we consider the n_0 when c is in position E . then, there are two diametrically opposite vertices associated with c on the same edge of the tessellation of order n_0 and these two vertices satisfy the positivity of the sums above.

Therefore,

$$\sup_{y, y' \in \mathcal{C}_c^N} |F(y') - F(y)| \geq C' 2^{-N\alpha}$$

where C' is a positive constant which only depends on α .

Finally, let $\tau \in (0, 1)$, $x \in [0, 1]^2$ and $N \geq 1$ such that $2^{-N} < \tau \leq 2^{-(N-1)}$. The ball $B_\tau(x)$ contains a cell \mathcal{C}_c^{N+1} of generation $N + 1$, thus

$$\text{osc}_\tau(x) \geq \sup_{y, y' \in \mathcal{C}_c^{N+1}} |F(y') - F(y)| \geq C' 2^{-(N+1)\alpha} \geq C'' \tau^\alpha,$$

which gives the lower bound in (38).

(ii) Combining (i) and Proposition 11.1 in [14] (see also [13]) we get $\mathcal{N}(\tau_N) \sim \tau_N^{\alpha-3}$ as $N \rightarrow \infty$. The result is then a consequence of (34). \square

3.2. Takagi-Knopp series generated by a random perturbation of the dyadic mesh.

In this subsection, we stray from the Voronoi partition of \mathbb{R}^D . An alternative way of randomizing the underlying partition of a Takagi-Knopp series is the following: the sequence of dyadic meshes $\mathcal{D}_n = 2^{-n}\mathbb{Z}^D$, $n \in \mathbb{N}$, is kept but each mesh \mathcal{D}_n is translated by a random uniform vector in $2^{-n}(0, 1)^D$. In each cube $C_{n,k}$, $k \in \mathbb{N}$, of \mathcal{D}_n , a random uniform ‘nucleus’ $c_{n,k}$ is chosen independently. The associated random pyramidal function Δ_n is defined so that it is equal to zero on \mathcal{D}_n and to 1 on the set $\{c_{n,k}, k \in \mathbb{N}\}$. The Takagi-Knopp type series $F_{\alpha,\beta}$, $\alpha, \beta > 0$, is then given by (2).

Compared to our Poisson-Voronoi construction, the main advantage of this model is that it preserves the cube structure so that it could be easier to deal with it in dimension two for applicational purpose in image analysis and pixel representation. The essential drawback is that the rigid structure of the mesh prevents us from obtaining an explicit Hausdorff dimension. Still, some results close to those proved in Section 1 can be deduced from similar methods. Indeed, if the random oscillation set $\mathcal{O}_{n,N}$, $n, N \in \mathbb{N}$, the increment $Z_n(x, y)$, $x, y \in \mathbb{R}^D$, the density z_n of $Z_n(x, y)$ conditionally on $\{x \in \mathcal{O}_{n,N}\}$ and the Lipschitz constant $L(x)$ are defined analogously, then the conclusions of Propositions 1.1 and 1.3 are satisfied. Moreover, the point (ii) of Proposition 1.2 is replaced by the following estimate:

$$\sup_{t \in \mathbb{R}} z_n(t) \leq \frac{C}{\mathbb{P}(x \in \mathcal{O}_{n,n})} \lambda^{-\frac{n(\beta-\alpha)}{D}} \sum_{i=1}^D \frac{1}{|x_i - y_i|}. \quad (40)$$

As a consequence, it is possible to derive an analogue of Proposition 1.4 and of the upper-bound of the box-dimension in (3). This upper-bound is in particular the exact Hausdorff dimension when $D = 1$. Indeed, in the linear case, the estimate of $\sup_{t \in \mathbb{R}} z_n(t)$ in (40) coincides with (ii) of Proposition 1.2, which implies that the lower-bound can be obtained along the same lines as in Section 2. We sum up our results in the next proposition, given without a detailed proof.

Proposition 3.2. *Let $F_{\alpha,\beta}$ be the function as above with $0 < \alpha \leq \beta \leq 1$. Then, its graph $\Gamma_{\alpha,\beta}$ satisfies almost surely the following estimates.*

- (i) *For every $D \geq 2$, $\dim_B(\Gamma_{\alpha,\beta}) \leq D + 1 - \frac{\alpha}{\beta}$.*
- (ii) *When $D = 1$, $\dim_B(\Gamma_{\alpha,\beta}) = \dim_H(\Gamma_{\alpha,\beta}) = 2 - \frac{\alpha}{\beta}$.*

Unfortunately, the fact that the sum $\sum_{i=1}^D \frac{1}{|x_i - y_i|}$ is a substitute for the inverse of the Euclidean norm $\|x - y\|^{-1}$ in the estimate (40) makes the proof of the lower-bound of the Hausdorff dimension more intricate for $D \geq 2$. This could be eventually considered as an extra-argument in favor of our Poisson-Voronoi construction.

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